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On the zeros of the derivatives of the Riemann zeta function under the Riemann hypothesis

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Abstract

The number of zeros and the distribution of the real part of non-real zeros of the derivatives of the Riemann zeta function have been investigated by Berndt, Levinson, Montgomery, and Akatsuka. Berndt, Levinson, and Montgomery investigated the general case, meanwhile Akatsuka gave sharper estimates for the first derivative of the Riemann zeta function under the truth of the Riemann hypothesis. In this report, we introduce a generalization of the results of Akatsuka to the k-th derivative (for positive integer k) of the Riemann zeta function.

1 Introduction

Zeros of the derivatives of the Riemann zeta function $\zeta(s)$ have been studied for about 80 years. In 1935, Speiser [Spe] showed that the Riemann hypothesis is equivalent to the first derivative of the Riemann zeta function $\zeta'(s)$ having no non-real zeros in $\mathrm{Re}(s) < 1/2$. This result is a breakthrough in the study of zeros of the Riemann zeta function. Following the work of Speiser, Spira [Spi65, Spi70] studied the zero-free regions of higher order derivatives of the Riemann zeta function, we write $\zeta^{(k)}(s)$ to denote the k-th derivative of the Riemann zeta function for positive integers k. These results encourage further study in the zeros of $\zeta^{(k)}(s)$. For example, in 1970, Berndt [Ber] investigated the number of zeros of $\zeta^{(k)}(s)$. He [Ber, Theorem] proved that for any positive integer k,

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.1}$$

holds, where $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ with $0 < \mathrm{Im}(s) \le T$, counted with multiplicity. Furthermore, in 1973, Spira [Spi73] also studied the relation between the zeros of $\zeta'(s)$ and the Riemann hypothesis. In 1974, Levinson and Montgomery [LM] studied many properties related to the distribution of zeros of $\zeta^{(k)}(s)$, including the location of zeros. They [LM, Theorem 10] also showed that for any positive integer k,

$$\begin{aligned}\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ \zeta^{(k)}(\rho^{(k)})=0,\, 0<\gamma^{(k)}\le T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T \\ - k\mathrm{Li}\left(\frac{T}{2\pi}\right) + O(\log T)\end{aligned} \tag{1.2}$$

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holds, where the sum is counted with multiplicity and

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

The above estimate shows how the real parts of non-real zeros of $\zeta^{(k)}(s)$ are distributed around the critical line $\mathrm{Re}(s) = 1/2$. The zeros of $\zeta^{(k)}(s)$ near the critical line were then studied further by Conrey and Ghosh [CG] in 1989.

In 1996, Yildirim [Yil96, Yil00] investigated non-real zeros of $\zeta''(s)$ and $\zeta'''(s)$ in the region to the left of the critical line, that is in the region $\mathrm{Re}(s) < 1/2$. He succeeded in showing that the Riemann hypothesis implies that $\zeta''(s)$ and $\zeta'''(s)$ each has only one pair of non-real zeros in $\mathrm{Re}(s) < 1/2$. Unfortunately, results analogous to Speiser's [Spe] were not obtained. Currently, no results similar to Speiser's [Spe] are known for higher order derivatives.

In 2012, Akatsuka [Aka, Theorems 1 and 3] improved each of the error term of the results obtained by Berndt and by Levinson and Montgomery mentioned above (eq. (1.1) and (1.2)) for the case $k = 1$ under the assumption of the truth of the Riemann hypothesis. He showed that

$$\sum_{\substack{\rho'=\beta'+i\gamma', \\ \zeta'(\rho')=0,\, 0<\gamma'\le T}} \left(\beta' - \frac{1}{2}\right) = \frac{T}{2\pi} \log\log \frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - \log\log 2\right) T - \mathrm{Li}\left(\frac{T}{2\pi}\right) + O((\log\log T)^2)$$

and

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O\left(\frac{\log T}{(\log\log T)^{1/2}}\right)$$

hold if the Riemann hypothesis is true. In this report, we are interested in investigating Akatsuka's method in the case when $k \ge 2$.

2 Some notation and main results

Before we introduce our results, we define some notation.

In this report we denote by $\mathbb{R}$ and $\mathbb{C}$ the set of all real numbers and the set of all complex numbers, respectively. Throughout this report, the letter k is used as a fixed positive integer, unless otherwise specified. For convenience, we let $\rho^{(k)} = \beta^{(k)} + i\gamma^{(k)}$ represent non-real zeros of $\zeta^{(k)}(s)$.

Each of the following results introduced in this report is a generalization of Theorem 1, Corollary 2, and Theorem 3 of [Aka], respectively. Note that each sum counts the non-real zeros of $\zeta^{(k)}(s)$ with multiplicity and that O_k denotes the error terms which depend only on k.

Theorem 1. *Assume that the Riemann hypothesis is true. Then for any $T > 4\pi$, we have*

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)}, \\ 0<\gamma^{(k)}\le T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi} \log\log \frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right) T$$

$$- k\mathrm{Li}\left(\frac{T}{2\pi}\right) + O_k((\log\log T)^2).$$

Corollary 2. *(Cf. [LM, Theorem 3].) Assume that the Riemann hypothesis is true. Then for $0 < U < T$ (where T is restricted to satisfy $T > 4\pi$), we have*

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ T<\gamma^{(k)}\leq T+U}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kU}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)U$$

$$+ O\left(\frac{U^2}{T\log T}\right) + O_k((\log\log T)^2).$$

Here the error term $O\left(\frac{U^2}{T\log T}\right)$ holds uniformly, in other words, it does not depend on any parameters.

Theorem 3. *Assume that the Riemann hypothesis is true. Then for $T \geq 2$, we have*

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + O_k\left(\frac{\log T}{(\log\log T)^{1/2}}\right),$$

where $N_k(T)$ is as defined in equation (1.1).

We write $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ (for any $s \in \mathbb{C}$) as σ and t respectively. We abbreviate the Riemann hypothesis as RH, and finally, we define two functions $F(s)$ and $G_k(s)$, as follows:

$$F(s) := 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s), \quad G_k(s) := (-1)^k\frac{2^s}{(\log 2)^k}\zeta^{(k)}(s).$$

By the above definition of $F(s)$, we can check easily that the functional equation for $\zeta(s)$ states

$$\zeta(s) = F(s)\zeta(1-s).$$

3 Sketch of proofs

In this report, we mainly give only sketch of the proofs. Refer to the original paper [Sur] for the details.

3.1 Key lemmas

We first introduce a few lemmas and propositions which are analogues of those in [Aka].

Lemma 3.1. *There exists an $a_k \geq 10$ such that*

$$|G_k(s) - 1| \leq \frac{1}{2}\left(\frac{2}{3}\right)^{\sigma/2}$$

holds for any $\sigma \geq a_k$.

Proof. See [LM, inequality (3.2) (p. 54)]. □

Lemma 3.2. *There exists a $\sigma_k \leq -1$ such that*

$$\left| \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} \frac{1}{\frac{F^{(k)}}{F^{(k-j)}}(s)} \frac{\zeta^{(j)}}{\zeta}(1-s) \right| \leq 2^{\sigma}$$

holds in the region $\sigma \leq \sigma_k, t \geq 2$.

Proof. (Sketch)
We begin by estimating

$$\frac{F^{(k)}}{F^{(k-j)}}(s) \quad (j = 1, 2, \cdots, k)$$

in the region $\sigma < 1, t \geq 2$. Using methods similar to [LM, pp. 54–55], we can show that for any positive integer k, we can take $\sigma_{k_1} \leq -1$ sufficiently small (i.e. sufficiently large in the negative direction) so that for any s with $\sigma \leq \sigma_{k_1}$ and $t \geq 2$, we have

$$\left| \frac{F^{(k)}}{F^{(k-j)}}(s) \right| \geq \frac{1}{2k} (\log (1-\sigma))^j. \tag{3.1}$$

Next we estimate

$$\frac{\zeta^{(j)}}{\zeta}(1-s) \quad (j = 1, 2, \cdots, k).$$

In the region $\sigma \leq -1, t \geq 2$, we can make use of the Dirichlet series of $\zeta(s)$ and $\zeta^{(j)}(s)$ to obtain

$$\left| \frac{\zeta^{(j)}}{\zeta}(1-s) \right| \leq \frac{2^{\sigma}}{2 - \frac{\pi^2}{6}} \left(\frac{1}{2} (\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l} \frac{j!}{(j-l)!}}{(-\sigma)^{l+1}} \right). \tag{3.2}$$

Now combining inequalities (3.1) and (3.2), for $\sigma \leq \sigma_{k_1}$ and $t \geq 2$, and noting that for any positive integer k,

$$\lim_{\sigma \to -\infty} \frac{2k}{2 - \frac{\pi^2}{6}} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{(\log (1-\sigma))^j} \left(\frac{1}{2} (\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l} \frac{j!}{(j-l)!}}{(-\sigma)^{l+1}} \right) = 0,$$

we can find $\sigma_k \leq \sigma_{k_1}$ (≤ -1) such that

$$\frac{2k}{2 - \frac{\pi^2}{6}} \sum_{j=1}^{k} \binom{k}{j} \frac{1}{(\log (1-\sigma))^j} \left(\frac{1}{2} (\log 2)^j + \sum_{l=0}^{j} \frac{(\log 2)^{j-l} \frac{j!}{(j-l)!}}{(-\sigma)^{l+1}} \right) \leq 1$$

holds for any $\sigma \leq \sigma_k$. This implies our lemma. □

Now we fix the above a_k and σ_k to show the following lemma.

Lemma 3.3. *Assume RH. Then there exists a $t_k \geq \max\{a_k^2, -\sigma_k\}$ such that the following conditions are satisfied:*

1. *For any s satisfying $\sigma_k \le \sigma \le 1/2$ and $t \ge t_k - 1$,*

$$\left|\frac{F^{(k)}}{F}(s)\right| \ge 1$$

holds. Furthermore, we can take a branch of $\log{(F^{(k)}/F)(s)}$ *in that region such that it is holomorphic there and*

$$\frac{\alpha_k \pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{\beta_k \pi}{6}$$

holds, where

$$(\alpha_k, \beta_k) = \begin{cases} (5,7) & \text{if } k \text{ is odd,} \\ (-1,1) & \text{if } k \text{ is even.} \end{cases}$$

2. *For any s satisfying $\sigma_k \le \sigma < 1/2$ and $t \ge t_k - 1$,*

$$\frac{\zeta^{(k)}}{\zeta}(s) \ne 0$$

holds. Furthermore, we can take a branch of $\log{(\zeta^{(k)}/\zeta)(s)}$ *in that region such that it is holomorphic there and*

$$\frac{k\pi}{2} < \arg \frac{\zeta^{(k)}}{\zeta}(s) < \frac{3k\pi}{2}$$

holds.

3. *For all $\sigma \in \mathbb{R}$, we have*

$$\zeta(\sigma + it_k) \ne 0,\ \zeta^{(k)}(\sigma + it_k) \ne 0.$$

Proof. (Sketch)
To prove condition 1, we apply Stirling's formula and methods similar to the proof of inequality (3.1). We can show that

$$F^{(k)}(s) = F(s)(-\log{(1-s)} + O(1))^k \left(1 + O\left(\frac{1}{|\log s|^2}\right)\right) \tag{3.3}$$

holds in the region $\sigma_k \le \sigma \le 1/2, t \ge 99$. Thus for any integer $k \ge 1$, we can take some $t_{k_1} \ge 100$ such that

$$\left|\frac{F^{(k)}}{F}(s)\right| \ge 1 \tag{3.4}$$

holds for $\sigma_k \le \sigma \le 1/2$ and $t \ge t_{k_1} - 1$.

We note from equation (3.3) that $(F^{(k)}/F)(s) = (-1)^k (\log t)^k + O((\log t)^{k-1})$ when $\sigma_k \le \sigma \le 1/2$ and $t \ge 99$. Consequently, for odd integer $k \ge 1$, we can find $t'_{k_2} \ge 100$ sufficiently large such that

$$\frac{5\pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{7\pi}{6}$$

holds for $\sigma_k \le \sigma \le 1/2$ and $t \ge t'_{k_2} - 1$. Similarly, when k is even, we can also find $t''_{k_2} \ge 100$ large enough such that

$$-\frac{\pi}{6} < \arg \frac{F^{(k)}}{F}(s) < \frac{\pi}{6}$$

holds for $\sigma_k \le \sigma \le 1/2$ and $t \ge t''_{k_2} - 1$. Since all zeros and poles of $F(s)$ lie on $\mathbb{R}$, $(F^{(k)}/F)(s)$ has no poles in $t > 0$. This along with inequality (3.4) implies that $\log (F^{(k)}/F)(s)$ is holomorphic in the region with this branch. We set

$$(\alpha_k, \beta_k) := \begin{cases} (5,7), & \text{if } k \text{ is odd,} \\ (-1,1), & \text{if } k \text{ is even;} \end{cases}$$

and

$$t_{k_2} := \begin{cases} t'_{k_2}, & \text{if } k \text{ is odd,} \\ t''_{k_2}, & \text{if } k \text{ is even.} \end{cases}$$

From the above calculations, we find that $\max\{t_{k_1}, t_{k_2}, a_k^2, -\sigma_k\}$ is a candidate for t_k. Thus we have proven that $t_k \ge \max\{a_k^2, -\sigma_k\}$ for which condition 1 holds exists. Since we want t_k to also satisfy conditions 2 and 3, we need to examine those conditions to completely prove the existence of t_k.

To prove condition 2, we first make use of the finiteness of the number of non-real zeros of $\zeta^{(j)}(s)$ in the region $\sigma < 1/2$ under RH for any positive integer j (cf. [LM, Corollary of Theorem 7 (p. 51)]) to find some t_{k_3} such that for all $j = 1, 2, \cdots, k$, we have

$$\zeta^{(j)}(s) \ne 0 \tag{3.5}$$

in the region $\sigma < 1/2, t \ge t_{k_3} - 1$.

Next we show that we can take a branch of $\log (\zeta^{(k)}/\zeta)(s)$ in the region $\sigma_k \le \sigma < 1/2, t \ge t_{k_4} - 1$ for some $t_{k_4} \ge 100$, so that it is holomorphic there and

$$\frac{k\pi}{2} < \arg \frac{\zeta^{(k)}}{\zeta}(s) < \frac{3k\pi}{2}$$

holds there by making use of the following inequality

$$\mathrm{Re}\left(\frac{\zeta^{(j)}}{\zeta^{(j-1)}}(s)\right) \le -\frac{2}{9}\log|s| + O_{\sigma_k}(1)$$

which holds for any $j = 1, 2, \cdots, k$ when $\sigma_k \le \sigma < 1/2$ and $t \ge t_{k_3} - 1$ (see [LM, pp. 64–65]). We omit details of the proof here.

We then have $\max\{t_{k_1}, t_{k_2}, t_{k_3}, t_{k_4}, a_k^2, -\sigma_k\}$ as a candidate for t_k.

Now we set $t_{k_5} := \max\{t_{k_1}, t_{k_2}, t_{k_3}, t_{k_4}, a_k^2, -\sigma_k\}$.

- Since we are assuming RH, $\zeta(\sigma + it) \ne 0$ for any $t > 0$ if $\sigma \ne 1/2$.
- According to [Spi65, Table 1 (p. 678) and Theorem 1], for any positive integer k, we have

$$\zeta^{(k)}(\sigma + it) \ne 0 \qquad (\sigma \ge 7k/4 + 2, t \in \mathbb{R}).$$

- Since $t_{k_5} \geq t_{k_3}$, from (3.5), we have $\zeta^{(k)}(\sigma + it) \neq 0$ for $\sigma < 1/2$ and $t \geq t_{k_5}$.

Hence, for any positive integer k, we only need to find $t_k \in [t_{k_5} + 1, t_{k_5} + 2]$ for which

$$\zeta\left(1/2 + it_k\right) \neq 0 \qquad \text{and} \qquad \zeta^{(k)}(\sigma + it_k) \neq 0 \quad \text{for } 1/2 \leq \sigma \leq 7k/4 + 2$$

hold.

Thus, we have shown that t_k defined above satisfies $t_k \geq \max\{a_k^2, -\sigma_k\}$ and also conditions 1 to 3. $\square$

We now fix a_k, σ_k, and t_k which satisfy Lemmas 3.1, 3.2, and 3.3.

Now we give two bounds for $-\arg\zeta(\sigma + iT) + \arg G_k(\sigma + iT)$. We use methods similar to [Aka, Lemmas 2.3, 2.4, 2.6]. We take the logarithmic branches of $\log\zeta(s)$ and $\log G_k(s)$ such that they tend to 0 as $\sigma \to \infty$ and are holomorphic in $\mathbb{C}\backslash\{\rho+\lambda \mid \zeta(\rho) = 0 \text{ or } \infty,\ \lambda \leq 0\}$ and $\mathbb{C}\backslash\{\rho^{(k)} + \lambda \mid \zeta^{(k)}(\rho^{(k)}) = 0 \text{ or } \infty,\ \lambda \leq 0\}$, respectively. We write

$$-\arg\zeta(\sigma + iT) + \arg G_k(\sigma + iT) = \arg\frac{G_k}{\zeta}(\sigma + iT),$$

where the argument on the right hand side is determined so that $\log\left(G_k/\zeta\right)(s)$ tends to 0 as $\sigma \to \infty$ and is holomorphic in $\mathbb{C}\backslash\{z + \lambda \mid (\zeta^{(k)}/\zeta)(z) = 0 \text{ or } \infty,\ \lambda \leq 0\}$.

Lemma 3.4. *Assume RH and let $T \geq t_k$. Then for any $\epsilon_0 > 0$ satisfying $\epsilon_0 < (2\log T)^{-1}$ (since $T \geq t_k \geq 100$, $\epsilon_0 < 1/8$), we have for $1/2 + \epsilon_0 < \sigma \leq a_k$,*

$$\arg\frac{G_k}{\zeta}(\sigma + iT) = O_{a_k,t_k}\left(\frac{\log\frac{\log T}{\epsilon_0}}{\sigma - \frac{1}{2} - \epsilon_0}\right).$$

We omit the proof of the above lemma (refer to [Sur, Lemma 2.3]).

Lemma 3.5. *Assume RH and let $A \geq 2$ be fixed. Then there exists a constant $C_0 > 0$ such that*

$$\left|\zeta^{(k)}(\sigma + it)\right| \leq \exp\left(C_0\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T} + (\log T)^{1/10}\right)\right)$$

holds for $T \geq t_k$, $T/2 \leq t \leq 2T$, $1/2 - (\log\log T)^{-1} \leq \sigma \leq A$.

Proof. Referring to [Tit, (14.14.2), (14.14.5) and the first equation on p. 384], we can show that

$$|\zeta(\sigma + it)| \leq \exp\left(C_1\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right) + (\log T)^{1/10}\right) \tag{3.6}$$

holds for $1/2 - 2(\log\log T)^{-1} \leq \sigma \leq A + 1$, $T/3 \leq t \leq 3T$ for some constant $C_1 > 0$ (cf. [Aka, pp. 2251–2252]).

Applying Cauchy's integral formula, we see that

$$\zeta^{(k)}(s) = \frac{k!}{2\pi i}\int_{|z-s|=\epsilon}\frac{\zeta(z)}{(z - s)^{k+1}}dz \qquad \text{for} \quad 0 < \epsilon < 1/2$$

holds in the region defined by $1/2 - (\log\log T)^{-1} \leq \sigma \leq A$ and $T/2 \leq t \leq 2T$. Applying inequality (3.6) and by taking $\epsilon = (2(\log\log T)^{1/k})^{-1}$ $(< 1/2)$, we obtain Lemma 3.5. $\square$

Lemma 3.6. *Assume RH and let $T \geq t_k$. Then for any $1/2 \leq \sigma \leq 3/4$, we have*

$$\arg G_k(\sigma + iT) = O_{a_k}\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right).$$

Proof. The proof proceeds in the same way as the proof of Lemma 2.4 of [Aka]. Refer to [Aka, pp. 2252–2253] for the detailed proof and use Lemma 3.5 above in place of Lemma 2.6 of [Aka]. □

Remark 1. The restrictions of the lower bound of T we gave in Lemmas 3.4, 3.5, and 3.6 are not essential, but they are sufficient for our needs.

3.2 Proof of Theorem 1

The following proposition gives the main term of Theorem 1.

Proposition 3.7. *Assume RH. Take a_k and t_k which satisfy Lemmas 3.1 and 3.3 respectively. Then for $T \geq t_k$ which satisfies $\zeta^{(k)}(\sigma + iT) \neq 0$ and $\zeta(\sigma + iT) \neq 0$ for any $\sigma \in \mathbb{R}$, we have*

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\leq T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T - k\mathrm{Li}\left(\frac{T}{2\pi}\right)$$
$$+ \frac{1}{2\pi}\int_{1/2}^{a_k} \left(-\arg\zeta(\sigma + iT) + \arg G_k(\sigma + iT)\right) d\sigma + O_k(1),$$

where the logarithmic branches are taken as in Section 3.1 (see the paragraph preceding Lemma 3.4).

We omit the proof (refer to [Sur, Proposition 2.2]). The proof of Theorem 1 is done as follows.

First of all, we consider for $T \geq t_k$ which satisfies $\zeta^{(k)}(\sigma + iT) \neq 0$ and $\zeta(\sigma + iT) \neq 0$ for any $\sigma \in \mathbb{R}$. From Lemma 3.4, we have

$$\int_{1/2+2\epsilon_0}^{a_k} \arg\frac{G_k}{\zeta}(\sigma + iT)d\sigma \ll_{a_k,t_k} \int_{1/2+2\epsilon_0}^{a_k} \frac{\log\frac{\log T}{\epsilon_0}}{\sigma - \frac{1}{2} - \epsilon_0}d\sigma \ll_{a_k} \log\frac{\log T}{\epsilon_0}\log\frac{1}{\epsilon_0}.$$

Next, from Lemma 3.6,

$$\arg G_k(\sigma + iT) = O_{a_k}\left(\frac{(\log T)^{2(1-\sigma)}}{(\log\log T)^{1/2}}\right) \qquad \text{for} \quad 1/2 \leq \sigma \leq 3/4$$

and from equation (2.23) of [Aka, p. 2251] (cf. [Tit, equations (14.14.3) and (14.14.5)]), RH implies that

$$\arg\zeta(\sigma + iT) = O\left(\frac{(\log T)^{2(1-\sigma)}}{\log\log T}\right)$$

holds uniformly for $1/2 \leq \sigma \leq 3/4$. Thus,

$$\int_{1/2}^{1/2+2\epsilon_0} \arg\frac{G_k}{\zeta}(\sigma + iT)d\sigma \ll_{a_k} \frac{\log T}{(\log\log T)^{1/2}}\epsilon_0.$$

Now we take $\epsilon_0 = (4\log T)^{-1}$ $(< (2\log T)^{-1})$, then we have

$$\int_{1/2}^{a_k} \arg \frac{G_k}{\zeta}(\sigma + iT) d\sigma \ll_{a_k,t_k} (\log\log T)^2.$$

Applying this to Proposition 3.7 and noting that a_k and t_k are fixed constants that depend only on k, we have

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{kT}{2\pi}\log\log\frac{T}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)T - k\mathrm{Li}\left(\frac{T}{2\pi}\right) + O_k((\log\log T)^2). \tag{3.7}$$

For $4\pi < T < t_k$, we are adding some finite number of terms which depend on t_k, and thus depend only on k so this can be included in the error term.

For $T \ge t_k$ such that $\zeta^{(k)}(\sigma + iT) = 0$ or $\zeta(\sigma + iT) = 0$ for some $\sigma \in \mathbb{R}$, there is some increment in the value of

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T}} \left(\beta^{(k)} - \frac{1}{2}\right)$$

as much as

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ \gamma^{(k)}=T}} \left(\beta^{(k)} - \frac{1}{2}\right).$$

We estimate this and show that this can be included in the error term of equation (3.7). We start by taking a small $0 < \epsilon < 1$ such that $\zeta^{(k)}(\sigma+i(T\pm\epsilon)) \ne 0$ and $\zeta(\sigma+i(T\pm\epsilon)) \ne 0$ for any $\sigma \in \mathbb{R}$. According to equation (3.7),

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ 0<\gamma^{(k)}\le T\pm\epsilon}} \left(\beta^{(k)} - \frac{1}{2}\right) = \frac{k(T\pm\epsilon)}{2\pi}\log\log\frac{T\pm\epsilon}{2\pi} + \frac{1}{2\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right)(T\pm\epsilon) - k\mathrm{Li}\left(\frac{T\pm\epsilon}{2\pi}\right) + O_k((\log\log T)^2).$$

Thus,

$$\begin{aligned}\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ T-\epsilon<\gamma^{(k)}\le T+\epsilon}} \left(\beta^{(k)} - \frac{1}{2}\right) &= \frac{k\epsilon}{\pi}\log\log\frac{T}{2\pi} + \frac{\epsilon}{\pi}\left(\frac{1}{2}\log 2 - k\log\log 2\right) + O_k((\log\log T)^2)\\ &= O_k((\log\log T)^2).\end{aligned}$$

This implies

$$\sum_{\substack{\rho^{(k)}=\beta^{(k)}+i\gamma^{(k)},\\ \gamma^{(k)}=T}} \left(\beta^{(k)} - \frac{1}{2}\right) = O_k((\log\log T)^2).$$

Therefore, this increment can also be included in the error term and the proof is complete. □

3.3 Proof of Corollary 2

This is an immediate consequence of Theorem 1. See [LM, p. 58 (the ending part of Section 3)]. $\square$

3.4 Proof of Theorem 3

Finally, we give the proof of Theorem 3. We first introduce the following proposition which give the main term of our estimate.

Proposition 3.8. *Assume RH. Take t_k which satisfies all conditions of Lemma 3.3. Then for $T \geq 2$ which satisfies $\zeta(\sigma + iT) \neq 0$ and $\zeta^{(k)}(\sigma + iT) \neq 0$ for all $\sigma \in \mathbb{R}$, we have*

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + \frac{1}{2\pi} \arg G_k \left(\frac{1}{2} + iT \right) + \frac{1}{2\pi} \arg \zeta \left(\frac{1}{2} + iT \right) + O_k(1),$$

where the arguments are determined as in Proposition 3.7.

We omit the proof (refer to [Sur, Proposition 3.1]). The proof of Theorem 3 is as follows.

Firstly we consider for $T \geq 2$ which satisfies $\zeta^{(k)}(\sigma + iT) \neq 0$ and $\zeta(\sigma + iT) \neq 0$ for any $\sigma \in \mathbb{R}$. By Lemma 3.6,

$$\arg G_k \left(\frac{1}{2} + iT \right) = O_{a_k} \left(\frac{\log T}{(\log \log T)^{1/2}} \right)$$

and again from equation (2.23) of [Aka, p. 2251], we have

$$\arg \zeta \left(\frac{1}{2} + iT \right) = O \left(\frac{\log T}{\log \log T} \right).$$

Substituting these into Proposition 3.8, we obtain

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + O_k \left(\frac{\log T}{(\log \log T)^{1/2}} \right).$$

Next, if $\zeta(\sigma + iT) = 0$ or $\zeta^{(k)}(\sigma + iT) = 0$ for some $\sigma \in \mathbb{R}$ $(T \geq 2)$, then again we take a small $0 < \epsilon < 1$ such that $\zeta^{(k)}(\sigma + i(T \pm \epsilon)) \neq 0$ and $\zeta(\sigma + i(T \pm \epsilon)) \neq 0$ for any $\sigma \in \mathbb{R}$ as in the proof of Theorem 1. Then similarly, we can show that the increment of the value of $N_k(T)$ can be included in the error term of the above equation which completes the proof. $\square$

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